A turbulent equatorial jet

By ROBERT R. LONG
Department of Mechanics, The Johns Hopkins University, Baltimore

(Received 31 January 1961 and in revised form 4 April 1961)

A recent study of a laminar jet in a rotating spherical shell of fluid is extended to the case of a turbulent planetary jet at the equator of a rotating, stratified atmosphere or ocean. General forms of the velocity, density and pressure functions of both the mean motion and the turbulence are derived by a dimensional analysis applied to the mean and perturbation equations. The horizontal and vertical dimensions are estimated, based on the three characteristic constants of the problem, which are the momentum transfer, the stability and a rotation parameter. The estimates are in good agreement with the dimensions of the Cromwell current, i.e. the equatorial undercurrent of the Pacific Ocean.

To the first order of approximation, the mean axial velocity in the theory is independent of distance along the jet axis. The mean horizontal transverse velocity component is much smaller and decreases upstream. The mean vertical velocity is extremely small, also decreasing upstream. The two horizontal velocity components of the turbulence are of the same order and, in the undercurrent, are about one-fourth the mean axial velocity. The vertical turbulent component is much smaller. Finally, it is shown that the eddy-viscosity concept is inappropriate for this problem because at least one of the eddy coefficients would have to be negative.

1. Introduction

In a recent paper (Long 1960) the author investigated a laminar, two-dimensional jet in a rotating spherical shell of homogeneous, viscous fluid. The flow is predominantly zonal, or west-east, and a suggested physical mechanism is a balance of vorticity brought into the jet by the weak north-south motion and diffused by friction. The order of magnitude of the horizontal width of the theoretical jet is given by $\overline{u}/\beta\delta_y^2 \sim 1$, where δ_y is the width, $\overline{u}$ is a representative velocity, and β is the north-south rate of change of the Coriolis parameter. This expression was used to estimate the width of the Cromwell current or equatorial undercurrent. This is a west-to-east current at the equator in the Pacific Ocean, located a hundred metres or so below the surface (Cromwell, Montgomery & Stroup 1954; Fofonoff & Montgomery 1955; Hidaka & Nagata 1958; Knauss & King 1958; see also the entire volume *Deep Sea Research*, **6**, 1960). If we use values $\overline{u} = 1{\cdot}5 \times 10^2\,\mathrm{cm\,sec^{-1}}$, $\beta = 2{\cdot}6 \times 10^{-13}\,\mathrm{cm^{-1}\,sec^{-1}}$, appropriate to the Cromwell current, we obtain $\delta_y \sim 2{\cdot}6 \times 10^7\,\mathrm{cm}$. This agrees well with the observed width, 300 km, as reported by Knauss (1960).

The laminar theory had two deficiences in relation to such geophysical jet flows as the undercurrent and the equatorial atmospheric jet (Lettau 1956): (1) the theory assumed that all quantities were independent of height, whereas the natural jets have much smaller thicknesses in the vertical than in the horizontal; (2) the motion was laminar, whereas the natural jets are no doubt fully turbulent. It was argued that the results could be applied to the undercurrent by introducing the concept of eddy viscosity, but the validity of this concept is always doubtful.

In this paper we investigate a fully turbulent jet in an infinite atmosphere or ocean. The axis of the jet is along the equator, and a general view is as pictured in figure 1. We may suppose that the flow is into a sink in the vicinity of $x = 0$, $y = 0$, $z = 0$. We confine attention, however, to the flow in the vicinity of $(-X_0, 0, 0)$ far upstream.†

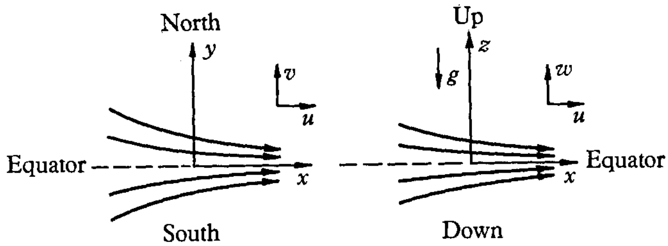

FIGURE 1. Schematic picture of mean motion in jet. The transverse velocity fields may have signs different from those suggested by this drawing.

2. Basic equations

We consider the fluid to be incompressible but inhomogeneous. At great distances vertically and horizontally from the jet axis, the gradient of density is linear in the vertical; i.e.

$$\frac{\partial \rho}{\partial z} \to -\frac{k\rho_0}{g}, \quad \frac{\partial \rho}{\partial x} \to 0, \quad \frac{\partial \rho}{\partial y} \to 0,$$

where k is a constant, g is gravity, and ρ_0 is a reference density. We also make the Boussinesq approximation (Boussinesq 1903) that density variations are negligible in their effects on the inertia of a parcel and in the equation of continuity, but of fundamental importance when they multiply gravity. Then, if we define a perturbation density ρ^* by

$$\rho = \rho_0 - \frac{kz}{g}\rho_0 + \rho^*,$$

and G, P by

$$G = -g\frac{\rho^*}{\rho_0}, \quad P = \frac{p}{\rho_0} + gz - \tfrac{1}{2}kz^2, \tag{1}$$

the equations of the problem can be written

$$u_t + uu_x + vu_y + wu_z - \beta yv = -P_x, \tag{2}$$

$$v_t + uv_x + vv_y + wv_z + \beta yu = -P_y, \tag{3}$$

$$w_t + uw_x + vw_y + ww_z - G = -P_z, \tag{4}$$

$$G_t + uG_x + vG_y + wG_z + kw = 0, \tag{5}$$

$$u_x + v_y + w_z = 0. \tag{6}$$

† We consider only westerly jets. Qualitative arguments, attributed to C. C. Lin, indicate that westerly jets are stable, easterly jets unstable to large-wavelength perturbations (Veronis 1960).

Equations (2) to (4) are the equations of motion, equation (5) expresses the incompressibility assumption, and (6) is the equation of continuity.

We have neglected molecular friction and diffusion. We know, of course, that these terms are important in so far as they affect the very small eddies, but it is well established that in fully developed turbulent flow at high Reynolds numbers, the effect on the mean motion as well as on the energy-containing eddies is negligible (Townsend 1956).

The terms $-\beta yv$ and βyu are the Coriolis accelerations, in which the Coriolis parameter $2\Omega \sin\theta$ (Ω is the angular rotation speed of the sphere, θ the latitude angle) has been approximated by $2\Omega y/a = \beta y$, where a is the radius of the sphere. This is a very good approximation if the north-south extent of the jet is sufficiently limited. Indeed, the error is of the same order as that involved in using equations in Cartesian instead of spherical co-ordinates, namely $o(\Lambda^{-2})$, where Λ is the ratio of the east-west and north-south length scales (Long 1960).

Finally we notice that in the two-dimensional theory (Long 1960), the solution described a jet motion that could exist at any latitude. In this paper the jet must be located at the equator if the equations are to have the form of (2)–(6).

We now take ensemble averages. Then, for example, we have

$$u = \bar{u} + u', \tag{7}$$

where $\bar{u}$ is the average velocity and $\overline{u'} = 0$. If we substitute into (2)–(6), we get a set of averaged and unaveraged differential equations:

$$\overset{\Lambda^{-\frac{1}{2}}}{\bar{u}\bar{u}_x} + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z - \beta y\bar{v} = -\overset{\Lambda^{-\frac{1}{2}}}{\bar{P}_x} - \overset{\Lambda^{-1}}{(\overline{u'u'})_x} - (\overline{u'v'})_y - (\overline{u'w'})_z, \tag{8}$$

$$\overset{\Lambda^{-2}}{\bar{u}\bar{v}_x} + \overset{\Lambda^{-2}}{\bar{v}\bar{v}_y} + \overset{\Lambda^{-2}}{\bar{w}\bar{v}_z} + \beta y\bar{u} = -\bar{P}_y - \overset{\Lambda^{-2}}{(\overline{u'v'})_x} - \overset{\Lambda^{-1}}{(\overline{v'v'})_y} - \overset{\Lambda^{-1}}{(\overline{v'w'})_z}, \tag{9}$$

$$\overset{\epsilon^2\Lambda^{-2}}{\bar{u}\bar{w}_x} + \overset{\epsilon^2\Lambda^{-2}}{\bar{v}\bar{w}_y} + \overset{\epsilon^2\Lambda^{-2}}{\bar{w}\bar{w}_z} - \bar{G} = -\bar{P}_z - \overset{\epsilon^2\Lambda^{-2}}{(\overline{u'w'})_x} - \overset{\epsilon^2\Lambda^{-1}}{(\overline{v'w'})_y} - \overset{\epsilon^2\Lambda^{-1}}{(\overline{w'w'})_z}, \tag{10}$$

$$\overset{\Lambda^{-\frac{1}{2}}}{\bar{u}\bar{G}_x} + \bar{v}\bar{G}_y + \bar{w}\bar{G}_z + \bar{w}k = -\overset{\Lambda^{-1}}{(\overline{u'G'})_x} - (\overline{v'G'})_y - (\overline{w'G'})_z, \tag{11}$$

$$\overset{\Lambda^{-\frac{1}{2}}}{\bar{u}_x} + \bar{v}_y + \bar{w}_z = 0, \tag{12}$$

and

$$u'_t + \bar{u}u'_x + \overset{\Lambda^{-\frac{3}{2}}}{u'\bar{u}_x} - \overset{\Lambda^{-\frac{3}{2}}}{(\overline{u'u'})_x} + \overset{\Lambda^{-\frac{1}{2}}}{u'u'_x} + \overset{\Lambda^{-1}}{\bar{v}u'_y} + v'\bar{u}_y + \overset{\Lambda^{-\frac{1}{2}}}{v'u'_y} - \overset{\Lambda^{-\frac{1}{2}}}{(\overline{u'v'})_y}$$
$$+ \overset{\Lambda^{-\frac{1}{2}}}{w'u'_z} + \overset{\Lambda^{-1}}{\bar{w}u'_z} + w'\bar{u}_z - \overset{\Lambda^{-\frac{1}{2}}}{(\overline{u'w'})_z} - \beta yv' = -P'_x, \tag{13}$$

$$v'_t + \overset{\Lambda^{-2}}{\bar{u}v'_x} + u'\bar{v}_x - \overset{\Lambda^{-\frac{3}{2}}}{(\overline{u'v'})_x} + \overset{\Lambda^{-\frac{1}{2}}}{u'v'_x} + \overset{\Lambda^{-1}}{\bar{v}v'_y} + \overset{\Lambda^{-1}}{v'\bar{v}_y} - \overset{\Lambda^{-\frac{1}{2}}}{(\overline{v'v'})_y} + \overset{\Lambda^{-\frac{1}{2}}}{v'v'_y}$$
$$+ \overset{\Lambda^{-\frac{1}{2}}}{w'v'_z} + \overset{\Lambda^{-1}}{\bar{w}v'_z} + \overset{\Lambda^{-1}}{w'\bar{v}_z} - \overset{\Lambda^{-\frac{1}{2}}}{(\overline{w'v'})_z} + \beta yu' = -P'_y, \tag{14}$$

$$\overset{\epsilon^2}{w'_t} + \overset{\epsilon^2}{\bar{u}w'_x} + \overset{\epsilon^2\Lambda^{-2}}{u'\bar{w}_x} - \overset{\epsilon^2\Lambda^{-\frac{3}{2}}}{(\overline{u'w'})_x} + \overset{\epsilon^2\Lambda^{-\frac{1}{2}}}{u'w'_x} + \overset{\epsilon^2\Lambda^{-1}}{\bar{v}w'_y} + \overset{\epsilon^2\Lambda^{-1}}{v'\bar{w}_y} - \overset{\epsilon^2\Lambda^{-\frac{1}{2}}}{(\overline{w'v'})_y} + \overset{\epsilon^2\Lambda^{-\frac{1}{2}}}{v'w'_y}$$
$$+ \overset{\epsilon^2\Lambda^{-1}}{w'\bar{w}_z} + \overset{\epsilon^2\Lambda^{-1}}{\bar{w}w'_z} - \overset{\epsilon^2\Lambda^{-\frac{1}{2}}}{(\overline{w'w'})_z} + \overset{\epsilon^2\Lambda^{-\frac{1}{2}}}{w'w'_z} - G' = -P'_z, \tag{15}$$

$$\overset{\Lambda^{-\frac{3}{4}}}{G'_t}+\overline{u}G'_x+\overset{\Lambda^{-\frac{3}{4}}}{u'\overline{G}_x}-(\overline{u'G'})_x+\overset{\Lambda^{-\frac{1}{2}}}{u'G'_x}+\overset{\Lambda^{-1}}{\overline{v}G'_y}+v'\overline{G}_y-\overset{\Lambda^{-\frac{1}{2}}}{(\overline{v'G'})_y}$$

$$+\overset{\Lambda^{-\frac{1}{2}}}{v'G'_y}+\overset{\Lambda^{-1}}{\overline{w}G'_z}+w'\overline{G}_z-\overset{\Lambda^{-\frac{1}{2}}}{(\overline{w'G'})_z}+\overset{\Lambda^{-\frac{1}{2}}}{w'G'_z}+kw' = 0, \quad (16)$$

$$u'_x+v'_y+w'_z = 0. \quad (17)$$

The symbols Λ^{-1}, $\epsilon^2\Lambda^{-1}$, etc., over the terms in these equations express the order of magnitude of the terms in a way which will be explained later.

If we now integrate equation (8) over a section of the jet, and assume that the velocities go to zero sufficiently fast at great distances from the jet axis, we get that

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\overline{P}+\overline{u}^2+\overset{\Lambda^{-1}}{\overline{u'u'}}-\tfrac{1}{2}\beta y^2\overline{u})\,dy\,dz \quad (18)$$

is constant, where we have used integration by parts and the equation of continuity to transform the Coriolis term. The constant J is fundamental to the problem. It is called the *momentum transfer*. If we specified the conditions at the sink precisely and all other conditions of the problem, and found the exact solution corresponding to zero motion at $|y| \to \infty$, $|z| \to \infty$, the integral (18) would be constant (independent of x), and J would be automatically determined. We confine attention, however, to the flow at large distances from the sink, and the solution we seek is not valid in the region (near the origin) where conditions are imposed that involve the precise description of the sink, say the flux Q and a linear dimension a. In adopting (18) as an alternative condition we must recognize the possibility that this may not be an adequate substitute for the two conditions involving a, Q, and that an indeterminacy may result. (Additional remarks about this point are contained below.)

3. General form of solution

We first make boundary-layer type estimates of the size of the terms in the mean equations. For example, it is plausible that the Coriolis and pressure terms alone dominate in (9), and that the vertical pressure variation is hydrostatic in (10). A first approximation for the unknowns in the equations of mean motion can then be obtained by using a generalized dimensional analysis, i.e. a search for invariance under a general affine transformation (Birkhoff 1950). We may then use the equations defining the Reynolds stresses to estimate the magnitude of such turbulent quantities as $u'v'$. A combination of these estimates of mean and turbulent quantities leads us to the general form of a first approximation to the solution and to a set of simplified differential equations that determine the first approximation. This in turn suggests a consistent scheme of successive approximations to the exact solution of the fully turbulent jet problem in the form of power series in certain small variables and parameters. We do not, to be sure, show that other expansions are not possible.

We first shift the origin of our co-ordinate system from the orifice to an origin at a great distance toward the west, X_0. Then x is the variable distance along the x-axis from the section in question. We introduce a typical velocity and length

$$c = \beta^{\frac{1}{7}}k^{\frac{1}{7}}J^{\frac{2}{7}}, \tag{19}$$

$$L = \beta^{-\frac{3}{7}}J^{\frac{1}{7}}k^{\frac{1}{14}}, \tag{20}$$

and non-dimensional independent variables

$$\xi = x/L, \quad \eta = y/L, \quad \zeta = z/L\epsilon, \quad \tau = tc/L, \tag{21}$$

where

$$\epsilon = J^{\frac{1}{7}}\beta^{\frac{4}{7}}k^{-\frac{3}{7}}. \tag{22}$$

We also need a non-dimensional function† of x:

$$\Lambda = (X_0 - x)/L. \tag{23}$$

We assume that Λ is very large. It is of the order of the ratio of the axial scale of the mean motion to the width of the jet if we assume $\eta \sim 1$. We see that if the quantities ζ and η in (21) are both of the order of one, ϵ is the ratio of the vertical and north-south length scales. We assume that ϵ is very small. (It is of the order of 10^{-3} in the case of the equatorial undercurrent.)

We now search for solutions for the mean and turbulent quantities in the forms‡

$$\overline{u}/c = U_1 + \Lambda^{-\frac{1}{2}}U_2 + \ldots + \epsilon^2 U_{21} + \ldots, \tag{24}$$

$$\overline{v}\Lambda/c = V_1 + \Lambda^{-\frac{1}{2}}V_2 + \ldots + \epsilon^2 V_{21} + \ldots, \tag{25}$$

$$\overline{w}\Lambda/\epsilon c = W_1 + \Lambda^{-\frac{1}{2}}W_2 + \ldots + \epsilon^2 W_{21} + \ldots, \tag{26}$$

$$\overline{P}/c^2 = P_1 + \Lambda^{-\frac{1}{2}}P_2 + \ldots + \epsilon^2 P_{21} + \ldots, \tag{27}$$

$$\overline{G}\epsilon L/c^2 = G_1 + \Lambda^{-\frac{1}{2}}G_2 + \ldots + \epsilon^2 G_{21} + \ldots, \tag{28}$$

$$\overline{u'u'}\Lambda/c^2 = F_1 + \Lambda^{-\frac{1}{2}}F_2 + \ldots + \epsilon^2 F_{21} + \ldots, \tag{29}$$

$$\overline{u'v'}\Lambda/c^2 = N_1 + \Lambda^{-\frac{1}{2}}N_2 + \ldots + \epsilon^2 N_{21} + \ldots, \tag{30}$$

$$\overline{u'w'}\Lambda/c^2\epsilon = H_1 + \Lambda^{-\frac{1}{2}}H_2 + \ldots + \epsilon^2 H_{21} + \ldots, \tag{31}$$

$$\overline{v'v'}\Lambda/c^2 = K_1 + \Lambda^{-\frac{1}{2}}K_2 + \ldots + \epsilon^2 K_{21} + \ldots, \tag{32}$$

$$\overline{v'w'}\Lambda/c^2\epsilon = L_1 + \Lambda^{-\frac{1}{2}}L_2 + \ldots + \epsilon^2 L_{21} + \ldots, \tag{33}$$

$$\overline{w'w'}\Lambda/c^2\epsilon^2 = M_1 + \Lambda^{-\frac{1}{2}}M_2 + \ldots + \epsilon^2 M_{21} + \ldots, \tag{34}$$

$$\overline{u'G'}\epsilon L\Lambda/c^3 = A_1 + \Lambda^{-\frac{1}{2}}A_2 + \ldots + \epsilon^2 A_{21} + \ldots, \tag{35}$$

$$\overline{v'G'}\epsilon L\Lambda/c^3 = B_1 + \Lambda^{-\frac{1}{2}}B_2 + \ldots + \epsilon^2 B_{21} + \ldots, \tag{36}$$

† Although we define Λ with the idea in mind that X_0 represents the distance to the orifice from the section at which we perform our analysis, in fact nothing in the analysis will be changed if we regard X_0 simply as a sufficiently large positive quantity with the dimensions of length. Some consequences of this are discussed below.

‡ The first approximations to the solution represent self-preserving flow, but in the development of this paper we *show* that this type of flow is possible at great distances from the orifice by considering all equations and conditions of the problem, not simply the equations of mean motion. Since the analysis may be applied to jets and wakes in homogeneous, non-rotating fluid, the approach of this paper removes the necessity for the customary *assumption* of self-preserving flow. The point is discussed by Townsend (1956).

$$\overline{w'G'}L\Lambda/c^3 = C_1+\Lambda^{-\frac{1}{2}}C_2+\ldots+\epsilon^2C_{21}+\ldots, \tag{37}$$

$$u'\Lambda^{\frac{1}{2}}/c = u_1+\Lambda^{-\frac{1}{2}}u_2+\ldots+\epsilon^2u_{21}+\ldots, \tag{38}$$

$$v'\Lambda^{\frac{1}{2}}/c = v_1+\Lambda^{-\frac{1}{2}}v_2+\ldots+\epsilon^2v_{21}+\ldots, \tag{39}$$

$$w'\Lambda^{\frac{1}{2}}/c\epsilon = w_1+\Lambda^{-\frac{1}{2}}w_2+\ldots+\epsilon^2w_{21}+\ldots, \tag{40}$$

$$P'\Lambda^{\frac{1}{2}}/c^2 = p_1+\Lambda^{-\frac{1}{2}}p_2+\ldots+\epsilon^2p_{21}+\ldots, \tag{41}$$

$$G'\epsilon L\Lambda^{\frac{1}{2}}/c^2 = g_1+\Lambda^{-\frac{1}{2}}g_2+\ldots+\epsilon^2g_{21}+\ldots, \tag{42}$$

where the functions denoted by capital letters, U_1, V_2, etc., are functions only of η, ζ, and those denoted by small letters, u_1, v_1, etc., are functions of ξ, η, ζ, τ.

If these expressions are substituted into equations (8)–(18), we find that the sets of quantities U_1, V_1–p_1, g_1; U_2, V_2–p_2, g_2; etc., satisfy sets of equations in independent variables ξ, η, ζ, τ, from which all the constants of the problem are missing and which do not involve Λ or ϵ. If we assume, therefore, that U_1, V_1–p_1, g_1 as well as ξ, η, ζ, τ are all of order one, we may then estimate the magnitude of all terms in equations (8)–(18). A quantity such as Λ^{-1}, $\epsilon^2\Lambda^{-2}$, etc., written over a term is the ratio of that term to the dominant terms of the equation.

The first set of equations is

$$V_1\frac{\partial U_1}{\partial\eta}+W_1\frac{\partial U_1}{\partial\zeta}-\eta V_1 = -\frac{\partial N_1}{\partial\eta}-\frac{\partial H_1}{\partial\zeta}, \tag{43}$$

$$\eta U_1 = -\frac{\partial P_1}{\partial\eta}, \tag{44}$$

$$-G_1 = -\frac{\partial P_1}{\partial\zeta}, \tag{45}$$

$$V_1\frac{\partial G_1}{\partial\eta}+W_1\frac{\partial G_1}{\partial\zeta}+W_1 = -\frac{\partial B_1}{\partial\eta}-\frac{\partial C_1}{\partial\zeta}, \tag{46}$$

$$\frac{\partial V_1}{\partial\eta}+\frac{\partial W_1}{\partial\zeta} = 0, \tag{47}$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(P_1+U_1^2-\frac{\eta^2U_1}{2}\right)d\eta\, d\zeta, \tag{48}$$

and

$$\frac{\partial u_1}{\partial\tau}+U_1\frac{\partial u_1}{\partial\xi}+v_1\frac{\partial U_1}{\partial\eta}+w_1\frac{\partial U_1}{\partial\zeta}-\eta v_1 = -\frac{\partial p_1}{\partial\xi}, \tag{49}$$

$$\frac{\partial v_1}{\partial\tau}+U_1\frac{\partial v_1}{\partial\xi}+\eta u_1 = -\frac{\partial p_1}{\partial\eta}, \tag{50}$$

$$-g_1 = -\frac{\partial p_1}{\partial\zeta}, \tag{51}$$

$$\frac{\partial g_1}{\partial\tau}+U_1\frac{\partial g_1}{\partial\xi}+v_1\frac{\partial G_1}{\partial\eta}+w_1\frac{\partial G_1}{\partial\zeta}+w_1 = 0, \tag{52}$$

$$\frac{\partial u_1}{\partial\xi}+\frac{\partial v_1}{\partial\eta}+\frac{\partial w_1}{\partial\zeta} = 0. \tag{53}$$

These are 10 differential equations in 14 unknowns U_1, V_1, W_1, P_1, G_1, u_1, v_1, w_1, p_1, g_1, N_1, H_1, B_1, C_1. We can make the system determinate, however, by adding the four equations

$$N_1 = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda u_1 v_1 d\xi, \tag{54}$$

$$H_1 = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda u_1 w_1 d\xi, \tag{55}$$

$$B_1 = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda v_1 g_1 d\xi, \tag{56}$$

$$C_1 = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda w_1 g_1 d\xi. \tag{57}$$

Since N_1, H_1, B_1, C_1 do not vary with ξ, we substitute for the ensemble average a space average along the x-axis. The equations (43)–(47), (49)–(53) and (54)–(57) are now a complete set of equations. Notice that if we multiply (52) by g_1 and average, we get

$$B_1\frac{\partial G_1}{\partial \eta}+C_1\left(\frac{\partial G_1}{\partial \zeta}+1\right) = 0. \tag{58}$$

This may be substituted for either (56) or (57).

The mathematical problem of the first approximation is not completely stated by the system of equations (43)–(57). Indeed, we see that if U_1, V_1, W_1, P_1, G_1, u_1, v_1, w_1, p_1, g_1, N_1, H_1, B_1, C_1 is a solution of the system, then there is an infinity of solutions U_1, α^2V_1, α^2W_1, P_1, G_1, αu_1, αv_1, αw_1, αp_1, αg_1, α^2N_1, α^2H_1, α^2B_1, α^2C_1, where α is any constant. Notice that as α is varied the solutions change in just the same way as a given solution changes when $X_0^{\frac{1}{2}}$ is varied. (The difference between X_0-x and X_0 in Λ is negligible for the purposes of the first approximation.) A similar indeterminacy arises in the non-rotating, homogeneous, turbulent jet. If the eddy-viscosity concept is used, it appears as an indeterminacy in the eddy Reynolds number (Townsend 1956).

One reason for the indeterminacy is that we have said nothing yet about our origin of time or about the turbulent velocity, pressure and density fields that exist at that instant. Something must be specified about those turbulent fields, but since we have assumed no variation with time of the mean quantities the initial fields are not completely arbitrary. For example, if we assume solutions for the turbulent quantities in the form

$$u_1 = u_{11}\cos(a\xi+b\tau)+u_{12}\sin(a\xi+b\tau),$$

$$v_1 = v_{11}\sin(a\xi+b\tau)+v_{12}\cos(a\xi+b\tau),$$

$$w_1 = w_{11}\sin(a\xi+b\tau)+w_{12}\cos(a\xi+b\tau),$$

$$p_1 = p_{11}\cos(a\xi+b\tau)+p_{12}\sin(a\xi+b\tau),$$

$$g_1 = g_{11}\cos(a\xi+b\tau)+g_{12}\sin(a\xi+b\tau),$$

where u_{11}, v_{12}, etc., are functions of η, ζ, and a, b are constants, we find from (54) to (57)

$$N_1 = \tfrac{1}{2}u_{11}v_{12} + \tfrac{1}{2}u_{12}v_{11},$$
$$H_1 = \tfrac{1}{2}u_{11}w_{12} + \tfrac{1}{2}u_{12}w_{11},$$
$$B_1 = \tfrac{1}{2}v_{11}g_{12} + \tfrac{1}{2}v_{12}g_{11},$$
$$C_1 = \tfrac{1}{2}w_{11}g_{12} + \tfrac{1}{2}w_{12}g_{11}.$$

On substitution into (43)–(53), the sine and cosine terms cancel out and we get 15 differential equations in 15 unknowns U_1, V_1, ..., u_{11}, v_{11}, ..., u_{12}, v_{12}, ... In this way we can arrive, in principle, at a solution of the first-approximation problem, but the initial conditions that are implied by this solution are very special. It is possible, of course, to assume solutions for the turbulent quantities that will satisfy more general initial conditions, but this increases the complexity of the equations that must be solved.

4. General properties of the jet

If we suppose that all the non-dimensional quantities that we have introduced (except Λ, ϵ) are of the order of 1, and if we confine attention to the first approximation in (24)–(42), we can state a number of properties of the jet:

(1) Combining $U_1 \sim 1$ and $\eta \sim 1$, we find that $\delta_y \sim \bar{u}^{\frac{1}{2}}\beta^{-\frac{1}{2}}$, where δ_y is the width of the jet. The width is small if the speed is low and if the rotation is high. In order of magnitude this is the same as the width of the laminar jet (Long 1960), although the laminar jet widens gradually with distance from the sink, whereas the width of the turbulent jet is uniform.

(2) Combining $\zeta \sim 1$, $\eta \sim 1$, we find that $\delta_z \sim \bar{u}k^{-\frac{1}{2}}$, where δ_z is the vertical thickness of the jet. Hence the vertical extent of the jet is small if the speed is low and the stability large. The vertical thickness is also independent of distance from the sink. Notice that the motion so adjusts itself that the non-dimensional numbers representing the rotation effect and the stability are of the order of one.

(3) The axial mean velocity is constant along the length of the jet.

(4) The transverse mean velocities both decrease upstream as $1/X_0$. In magnitude, $\bar{v} \ll \bar{u}$ and $\bar{w} \ll \bar{v}$.

(5) The mean-density field is independent of distance along the axis.

(6) The two horizontal turbulent velocity components are of the same order and smaller than the axial mean velocity. The vertical and horizontal transverse turbulent velocities are larger than the corresponding mean quantities. The vertical turbulent velocity is much smaller than the horizontal components.

(7) All turbulent quantities decrease rather gradually upstream as $X_0^{-\frac{1}{2}}$.

(8) The horizontal scale of the turbulence is of the order of the horizontal width of the jet; the vertical scale of the turbulence is of the order of the thickness of the jet.

An important result may be obtained concerning the eddy-viscosity concept applied to this problem. For example, if we assume the turbulent transports are proportional to the mean density gradients

$$\overline{v'G'} = -K_y\frac{\partial \bar{G}}{\partial y}, \quad \overline{w'G'} = -K_z\left(\frac{\partial \bar{G}}{\partial z} + k\right),$$

where K_y, K_z are horizontal and vertical components of eddy diffusivity, the form of (46) is preserved only if

$$K_y = \frac{1}{S_y}\Lambda^{-1}\beta^{-\frac{2}{7}}k^{\frac{3}{14}}J^{\frac{3}{7}}, \quad B_1 = -\frac{1}{S_y}\frac{\partial G_1}{\partial \eta},$$

$$K_z = \frac{1}{S_z}\Lambda^{-1}\beta^{\frac{6}{7}}k^{-\frac{9}{14}}J^{\frac{5}{7}}, \quad C_1 = -\frac{1}{S_z}\left(\frac{\partial G_1}{\partial \zeta}+1\right),$$

where S_y, S_z are non-dimensional functions of η, ζ. The coefficients of diffusion therefore decrease upstream as we would expect, and in general vary with the constants of the problem in a plausible way. But using (58) we have

$$\frac{1}{S_y}\left(\frac{\partial G_1}{\partial \eta}\right)^2 + \frac{1}{S_z}\left(\frac{\partial G_1}{\partial \zeta}+1\right)^2 = 0.$$

This can only be true, however, if the diffusion coefficients are allowed to be negative. We see clearly that an analogy with molecular diffusion and friction is not useful in this problem of turbulence.

5. Circumferential equatorial jets

In the next section we apply the theory of the turbulent jet to the equatorial undercurrent. Before doing this we may mention here that westerly currents at the equator seem to be a very common feature of the atmospheres of the planets and the sun. In astrophysics it is known as the equatorial acceleration (Wasiutyński 1946; Cowling 1953; Hess 1951). In Jupiter, for example, this acceleration occurs in a band between $-7°$ and $+7°$ latitude and amounts to a relative angular velocity of about 1 % of the angular velocity of the planet ($\omega/\Omega \sim 0{\cdot}01$). In the atmosphere of the sun the acceleration is very strong, but very variable within a sunspot cycle, ω/Ω varying from 0·3 to 0·7 or so. The band containing high-speed fluid is between $-55°$ and $+55°$ when ω/Ω is a minimum; when ω/Ω is a maximum, the width of the band is very indefinite, but certainly even broader than this.

The same phenomenon occurs in our own atmosphere, although it is not yet known whether it extends round the entire globe (Lettau 1956). This jet is called the Berson westerlies by Lettau in honour of Berson who first reported the phenomenon (1910). Observations indicate that it is located in the stratosphere at around 20 km. It is about 5 km thick and 1500 km in north-south extent. The speeds are of the order of 10^3 cm sec^{-1}.

The theory of this paper can be modified to apply to this phenomenon. We need only recognize that there can be no variation of the mean quantities with longitude, so that Λ is simply an undetermined large number. The order of the width of the jet is the same, $\delta_y \sim \bar{u}^{\frac{1}{2}}\beta^{-\frac{1}{2}}$, or $\delta\theta \sim (\omega/2\Omega)^{\frac{1}{2}}$. This seems to describe in a rough way the variation in the size of the jet in the astrophysical examples.

If we apply the theory to the Berson westerlies, we get $\delta_y \sim 6 \times 10^7$ cm. This is considerably less than the observed width, but our knowledge of this current is still quite limited. If we use $\bar{u} = 10^3$, $k = 4 \times 10^{-4}$ we find $\delta_z \sim 5 \times 10^4$ cm which is very much too small. However, we should note that the westerly jet is imbedded in a belt of easterlies with a general drift of perhaps 3×10^3 cm sec^{-1}. It is more reasonable then to use for $\bar{u}$ the relative velocity of 4×10^3. This yields $\delta_z \sim 2 \times 10^5$ cm and $\delta_y \sim 1{\cdot}2 \times 10^8$ cm.

6. Application to the equatorial undercurrent

In any attempt to apply the above theory to the equatorial undercurrent we must recognize that an ocean surface exists in one case and not in the other. Our theory of a jet in an infinite fluid can be adapted to this case, however, as follows: if we guess that the mean theoretical axial velocity profile in a vertical section resembles that in figure 2 (counter currents are typical phenomena in rotating and stratified fluids), we could put in a free surface S at the indicated position without changing the situation substantially, provided the surface stress in the natural phenomenon is substituted for the stress exerted by the fluid above S in the theoretical situation. If we adopt this viewpoint, this force, supplied by the wind stress in the case of the ocean, may be related to the depth of the jet axis below the surface.

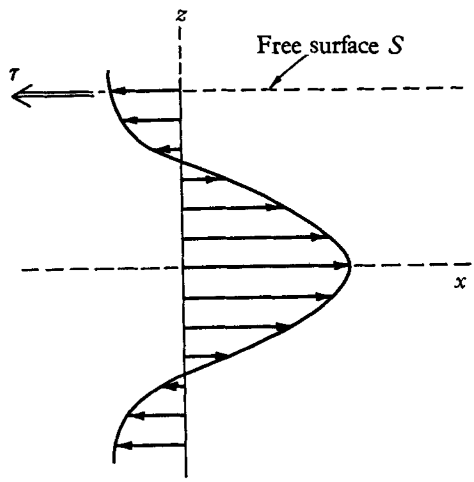

FIGURE 2. Schematic picture of vertical velocity profile of undercurrent.

In applying our theory to the Cromwell current we take as observed values (Knauss 1960) a mean velocity of $1{\cdot}5 \times 10^2$ cm sec^{-1}, width 3×10^7 cm, vertical thickness 2×10^4 cm, longitudinal length 5×10^8 cm or more, $\beta = 2{\cdot}3 \times 10^{-13}$ cm^{-1} sec^{-1}. If we use the given values of $\bar{u}$ and β, we may compute δ_y and δ_z. We get $\delta_y \sim 2{\cdot}6 \times 10^7$ cm, $\delta_z \sim 10^4$ cm, where we have estimated the stability parameter† $k = (g/\rho)\, d\rho/dz \sim 4 \times 10^{-4}$ sec^{-2}. Since the estimates are only supposed to be order of magnitude estimates, these results are remarkably close to the mark. The value of Λ is about 20 in this case, so that all turbulent quantities are $\frac{1}{4}$ to $\frac{1}{5}$ of the corresponding mean quantities.

Although we do not solve for the mean velocity and temperature fields in the jet we may make certain inferences about them. In figure 3 we picture the flow

† This estimate was supplied to me by Professor R. B. Montgomery.

in a vertical north-south cross-section. Since the motion is westerly in the jet the y- and z-equations of mean motion (or the thermal wind equation) require a temperature field in which the isotherms are roughly as drawn. This suggests the vertical and horizontal transverse velocities shown, and this is consistent with the equation of continuity (47). It is important to remark that this is precisely the observed form of the constant density surfaces in the vicinity of the Cromwell current as we see in figure 4. This transverse velocity field is also consistent with the Ekman wind-drift theory, which requires that the surface water move north and south away from the equator in this region.

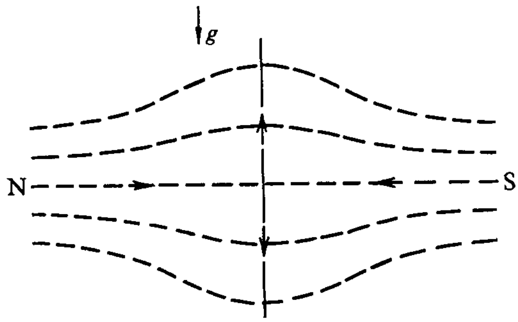

FIGURE 3. Mean temperature distribution in jet.

The author wishes to thank Professor R. B. Montgomery and Mr R. S. Arthur for a number of stimulating conversations about the undercurrent. This research was sponsored by the Office of Naval Research and the U.S. Weather Bureau.

Appendix

Jets are very common in stratified and rotating fluid systems. For example if a rotating fluid is heated, very pronounced, irregular jets occur at the free surface (Fultz *et al.* 1959). They resemble the famous Jet Stream of the atmosphere and other currents like the Gulf Stream in the oceans. In these phenomena the rotation itself, rather than the variation of the rotation, is probably the more important effect. An analysis, essentially the same as that above, can be applied to such a jet, using the same equations as (2)–(6) except that $f = 2\Omega$ replaces βy. The solutions all have the same form as (24)–(42). However, now we have $c = J^{\frac{1}{4}}f^{\frac{1}{4}}k^{\frac{1}{8}}$, $L = J^{\frac{1}{4}}f^{-\frac{3}{4}}k^{\frac{1}{8}}$, $\epsilon = fk^{-\frac{1}{2}}$, and in the differential equations of the first approximation η is missing in the terms corresponding to the Coriolis forces. We may compare these results with observations of the Jet Stream and Gulf Stream (Newton 1959; Stommel 1960). For example, Newton's data give for the ratio of depth to width 0·005 for both. We get precisely 0·005 for ϵ if we use the reasonable values $f = 10^{-4}$, $k = 4 \times 10^{-4}$. On the other hand, the theoretical estimates of the dimensions of the Jet Stream and Gulf Stream are too low by a factor of 2 or so.

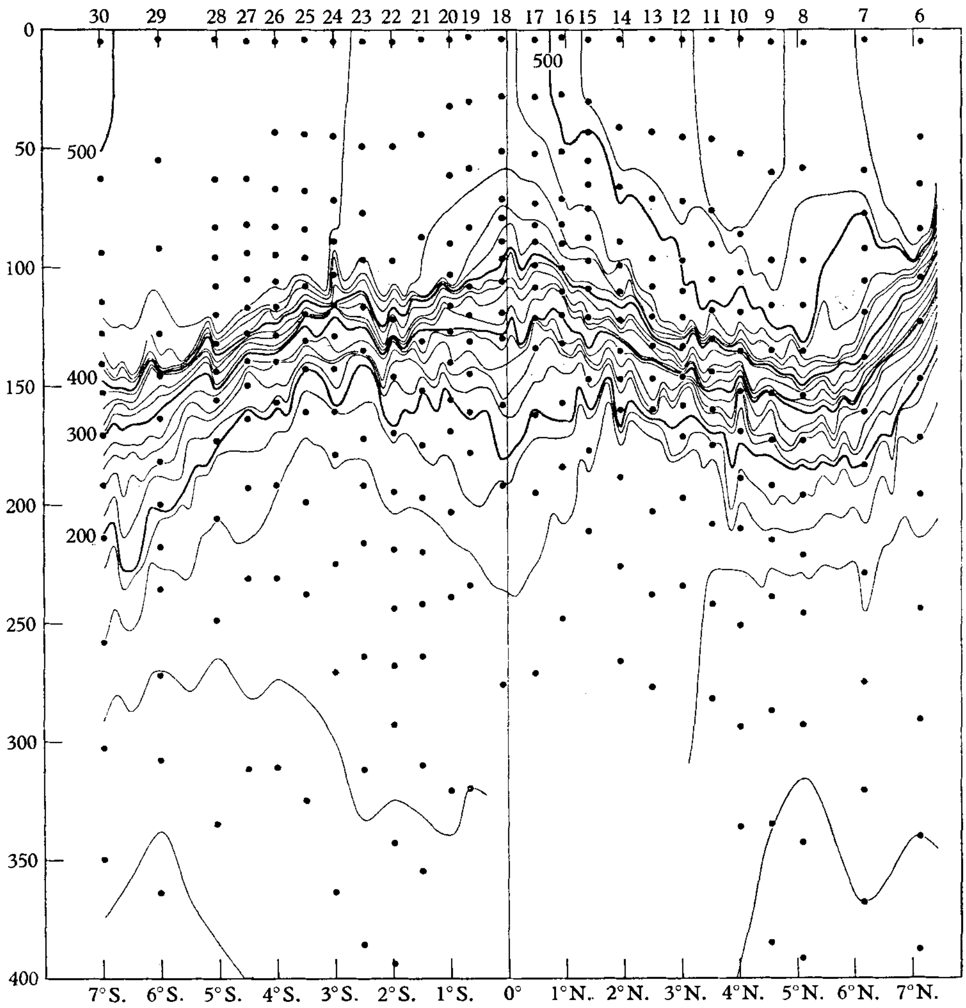

FIGURE 4. Specific-volume anomalies in centilitres per ton on vertical section in Pacific Ocean at 150° W. in July–August 1952. Observations from *Hugh M. Smith* of Pacific Ocean Fishery Investigations, U.S. Fish and Wildlife Service. Drawing from unpublished manuscript by R. B. Montgomery and E. D. Stroup.

REFERENCES

BERSON, A. 1910 Bericht über die aerologische Expedition des Königl. Aeronautischen Observatorium nach Ostafrika im Jahre 1910. *Ergebn. Arb. Preuz. Aeronaut. Obs. Lindenberg: Braunschweig.*

BIRKHOFF, G. 1950 *Hydrodynamics*, chapter IV. Princeton University Press.

BOUSSINESQ, J. 1903 *Théorie analytique de la chaleur*, Vol. **2**, p. 172. Gauthier-Villars.

COWLING, T. G. 1953 Solar electrodynamics. Article in *The Sun* (Ed. G. P. Kuiper). University of Chicago Press.

CROMWELL, T., MONTGOMERY, R. B. & STROUP, E. D. 1954 Equatorial undercurrent in Pacific Ocean revealed by new methods. *Science*, **119**, 648.

Fofonoff, N. P. & Montgomery, R. B. 1955 The equatorial undercurrent in the light of the vorticity equation. *Tellus*, **4**, 518.

Fultz, D., Long, R. R., Owens, G. V., Bohan, W., Kaylor, R. & Weil, J. 1959 Studies of thermal convection in a rotating cylinder with some implications for large-scale atmospheric motions. *Meteor. Monograph*, **4**.

Hess, S. L. 1951 The atmospheres of the other planets. *The Compendium of Meteorology*, p. 391.

Hidaka, K. & Nagata, Y. 1958 Dynamical computation of the equatorial current system of the Pacific, with special application to the equatorial undercurrent. *Geophys. J. Roy. Astr. Soc.* **1**, 198.

Knauss, J. A. 1960 Measurements of the Cromwell Current. *Deep Sea Res.* **6**, 265.

Knauss, J. A. & King, J. F. 1958 Observations of the Pacific equatorial undercurrent. *Nature*, **182**, 601.

Lettau, H. 1956 Theoretical notes on the dynamics of the equatorial atmosphere. *Beitr. Phys. Atm.* **29**, 107.

Long, R. R. 1960 A laminar planetary jet. *J. Fluid Mech.* **7**, 632.

Newton, C. W. 1959 Synoptic comparison of Jet Stream and Gulf Stream systems. *The Atmosphere and Sea in Motion*, p. 288. New York: The Rockefeller Institute Press.

Stommel, H. 1960 *The Gulf Stream*, p. 202. University of California Press.

Townsend, A. A. 1956 *The Structure of Turbulent Shear Flow*, p. 89. Cambridge University Press.

Veronis, G. 1960 An approximate analysis of the equatorial undercurrent. *Deep Sea Res.* **6**, 318.

Wasiutyński, J. 1946 Hydrodynamics and structure of stars and planets. *Astrophysica Norvegica*, **4**, 497.